JOURNAL OF APPROXIMATION THEORY 80, 409–422 (1995)

On a Problem of N. Kirchoff and R. J. Nessel

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Communicated by Paul L. Butzer

Received April 30, 1993; accepted August 17, 1993

This paper is devoted to the solution of a problem of N. Kirchoff and R. J. Nessel on the existence of a function $f \in C_{2\pi}$ such that

$$\limsup_{n \to \infty} \frac{|F_n f(x) - f(x)|}{|T_n f(x) - f(x)|} = +\infty$$

for almost all $x \in R$, where F_n is the trigonometric convolution operator and T_n is its discrete analogue.

Let $C_{2\pi}$ be the Banach space of functions f, 2π-periodic and continuous on the real axis R, endowed with the usual sup-norm $\|f\|_c := \sup\{|f(u)| : u \in R\}$.

For an even polynomial kernel of degree n, $n \in N$ (set of natural numbers), given by

$$X_n(x) := \sum_{k=-n}^{n} \rho_{k,n} e^{ikx} \tag{1}$$

with $\rho_{-k,n} = \rho_{k,n}$, $\rho_{0,n} = 1$, and for $f \in C_{2\pi}$ let the trigonometric convolution operator be defined by

$$F_n f(x) := \frac{1}{2\pi} \int_0^{2\pi} f(u)\, X_n(x-u)\, du \tag{2}$$

and its discrete analogue by ($u_{j,n} = 2\pi j/2n+1$, $0 \leqslant j \leqslant 2n$)

$$T_n f(x) := \frac{1}{2n+1} \cdot \sum_{j=0}^{2n} f(u_{j,n})\, X_n(x - u_{j,n}). \tag{3}$$

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0021-9045/95 $6.00

For $h_k(x) := e^{ikx}$, $k \in Z$ (set of integers), one has

$$F_n h_k(x) = \rho_{k,n} h_k(x) = T_n h_k(x) \qquad (|k| \leqslant n). \tag{4}$$

For the relations between operators F_n and T_n see [3, 4].

From the results of N. Kirchoff and R. J. Nessel (cf. [2, p. 35]) it follows that if

$$\|X_n\|_1 := \int_0^{2\pi} |X_n(u)|\, du = O(1) \qquad (n \to \infty) \tag{5}$$

and

$$1 - \rho_{j,n} = O_j\left(\frac{1}{n}\right) \qquad (j \in N,\, n \to \infty), \tag{6}$$

then there exists a counterexample $f_0 \in C_{2\pi}$ such that

$$\limsup_{n \to \infty} \frac{|T_n f_0(x) - f_0(x)|}{|F_n f_0(x) - f_0(x)|} = +\infty \tag{7}$$

for almost every $x \in R$.

In the proof of this result use is made of the following extension of Calderon's lemma (cf. [5, p. 165]).

THEOREM (N. Kirchoff and R. J. Nessel [2, p. 30]). *Let H_k, $D_k \subset R$ be (Lebesgue) measurable subsets such that H_k is 2π-periodic and D_k belongs to $[0, 2\pi]$ with Lebesgue measure $\mu(D_k) \neq 0$ for each $k \in N$. Suppose that*

$$\left\| \prod_{k=1}^{n} \left(1 - \frac{\mu(D_k \cap (H_k - t))}{\mu(D_k)}\right)\right\|_1 = o(1) \qquad (n \to \infty). \tag{8}$$

Then there exist points $y_k \in D_k$ such that $\limsup_{k \to \infty} (H_k - y_k) := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (H_k - y_k)$ is a set of full measure.

In [2, p. 38] is posed the problem on the existence (under the conditions (5) and (6)) of a counterexample $f \in C_{2\pi}$ such that

$$\limsup_{n \to \infty} \frac{|F_n f(x) - f(x)|}{|T_n f(x) - f(x)|} = +\infty$$

for almost every $x \in R$.

The present paper is devoted to the solution of this problem. Namely, we shall prove the following

THEOREM. *Let* (5) *and* (6) *hold. Then there exists a* (*real-valued*) *counterexample* $f \in C_{2\pi}$ *such that*

$$\limsup_{n \to \infty} \frac{|F_n f(x) - f(x)|}{|T_n f(x) - f(x)|} = +\infty \tag{9}$$

for almost every $x \in R$.

First we shall prove a number of lemmas.

LEMMA 1. *Let* (5) *and* (6) *hold. Then*

$$\lim_{n \to \infty} \|X_n\|_c = +\infty. \tag{10}$$

Proof. It is clear that (cf. (1))

$$\|X_n\|_2^2 := \int_0^{2\pi} X_n^2(u)\, du = 1 + 2 \sum_{k=1}^{n} \rho_{k,n}^2.$$

Then from (6) we have

$$\lim_{n \to \infty} \|X_n\|_2^2 = +\infty. \tag{11}$$

But according to (5) we obtain

$$\|X_n\|_2^2 \leqslant \|X_n\|_c \cdot \|X_n\|_1 = O(\|X_n\|_c).$$

Now taking account of (11) we conclude that Lemma 1 is proved.

LEMMA 2. *Let* (5) *and* (6) *hold. Then for any* $n \in N$ *there exists a number* $\alpha \in [0, 2\pi)$ *such that*

$$\left(\alpha, \alpha + \frac{1}{2n}\right) \subset [0, 2\pi) \tag{12}$$

and for any $u \in (\alpha, \alpha + 1/2n)$ *one has*

$$|X_n(u)| > \tfrac{1}{2} \|X_n\|_c. \tag{13}$$

Proof. Let $x_0 \in [0, 2\pi)$ be a point such that

$$|X_n(x_0)| = \|X_n\|_c. \tag{14}$$

Without loss of generality we may assume that $X_n(x_0) > 0$. According to the theorems of Lagrange and Bernstein, if $|h| \in (0, 1/2n)$, then there exists a number $\xi \in (0, |h|)$ such that

$$|X_n(x_0+h) - X_n(x_0)| = |X_n^1(\xi)| \cdot |h| \leqslant \|X_n^1\|_c \cdot |h|$$
$$\leqslant n \|X_n\|_c \cdot |h| \leqslant \tfrac{1}{2} \|X_n\|_c.$$

Consequently, if $|h| \in (0, 1/2n)$, then (cf. (14))

$$X_n(x_0+h) \geqslant X_n(x_0) - \tfrac{1}{2} \|X_n\|_c = \tfrac{1}{2} \|X_n\|_c.$$

It is obvious that either $x_0 + 1/2n \in [0, 2\pi)$ or $x_0 - 1/2n \in [0, 2\pi)$ and thus Lemma 2 is proved.

For convenience we shall use a notation

$$w(n) := \|X_n\|_c, \qquad n = 1, 2, ... \tag{15}$$

We introduce the sets $(n > n_0)$

$$E_n = \bigcup_{j=0}^{2n} \left(\frac{2\pi j}{2n+1} + \frac{1}{2n\sqrt{w(n)}}, \frac{2\pi(j+1)}{2n+1} \right), \tag{16}$$

where $n_0 > 2$ is chosen such that (cf. (10), (15))

$$w(n) > (64\pi)^2 \qquad (n > n_0). \tag{17}$$

LEMMA 3. *Let* (5) *and* (6) *hold. Then for any* $n \geqslant n_0$ *there exist a real-valued trigonometric polynomial* $P_n(x)$ *and a set* $A_n \subset [0, 2\pi]$ *such that* (*cf.* (2), (15), (16))

$$\|P_n\|_c \leqslant 2, \tag{18}$$

$$A_n \subset E_n, \tag{19}$$

$$\mu A_n \geqslant \gamma_1, \tag{20}$$

$$|F_n P_n(x) - P_n(x)| \geqslant \frac{C_1 \sqrt{w(n)}}{n} \qquad (x \in A_n), \tag{21}$$

and

$$|P_n(x)| \leqslant \frac{1}{nw(n)} \qquad (x \in E_n), \tag{22}$$

where n_0, C_1, *and* γ_1 *are positive constants.*

Proof. Without loss of generality we may assume that the number α from Lemma 2 satisfies the condition

$$0<\alpha<\pi. \tag{23}$$

Let j_0 be an integer such that

$$\frac{2\pi(j_0-2)}{2n+1}-\alpha\leqslant 0<\frac{2\pi(j_0-1)}{2n+1}-\alpha. \tag{24}$$

Taking account of (24) and (23) we obtain

$$1<j_0<n+3. \tag{25}$$

We introduce the sets $(j=0, 1, ..., 2n)$

$$B_n^{(j)}:=\left(\frac{2\pi j}{2n+1}+\frac{1}{2n\sqrt{w(n)}}-\alpha-\frac{1}{2n}, \frac{2\pi j}{2n+1}-\alpha\right), \tag{26}$$

$$C_n^{(j)}:=\left[\frac{2\pi j}{2n+1}, \frac{2\pi j}{2n+1}+\frac{1}{2n\sqrt{w(n)}}\right). \tag{27}$$

Let $f_n^{(j)}(x)$ $(j=j_0, ..., 2n)$ be 2π-periodic function defined by the following equality (cf. (27))

$$f_n^{(j)}(x)=\begin{cases}1 & \text{for } x\in C_n^{(j)}\\ 0 & \text{for } x\in[0, 2\pi)\backslash C_n^{(j)}.\end{cases} \tag{28}$$

It is easy to see that for $n>n_0$ we have (cf. (17), (24)–(28))

$$\operatorname{supp} f_n^{(p)}(x)\cap \operatorname{supp} f_n^{(q)}(x)=\varnothing \quad \text{for } j_0\leqslant p\neq q\leqslant 2n, \tag{29}$$

$$B_n^{(p)}\cap B_n^{(q)}=\phi \quad \text{for } j_0\leqslant p\neq q\leqslant 2n, \tag{30}$$

$$B_n^{(j)}\subset[0, 2\pi) \quad \text{for } j_0\leqslant j\leqslant 2n, \tag{31}$$

$$C_n^{(j)}\subset[0, 2\pi) \quad \text{for } j_0\leqslant j\leqslant 2n. \tag{32}$$

Consider the set (cf. (26) and (24))

$$B_n:=\bigcup_{j=j_0}^{2n} B_n^{(j)}. \tag{33}$$

Then from (33), (26), (30), (25), (17) we have $(n>n_0)$

$$\mu B_n=(2n-j_0+1)\left(\frac{1}{2n}-\frac{1}{2n\sqrt{w(n)}}\right)\geqslant\frac{1}{4}. \tag{34}$$

We shall show that if $x \in B_n^{(i_0)}$ for some i_0, $j_0 \leqslant i_0 \leqslant 2n$, then

$$|F_n f_n^{(i_0)}(x)| > \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n}. \qquad (35)$$

Indeed (cf. (2), (28)) we have

$$F_n f_n^{(i_0)}(x) = \frac{1}{2\pi} \cdot \int_{C_n^{(i_0)}} X_n(u-x)\, du. \qquad (36)$$

Further we note that when $u \in C_n^{(i_0)}$ and $x \in B_n^{(i_0)}$ (cf. (26), (27)), then

$$u - x \in (\alpha, \alpha + 1/2n). \qquad (37)$$

According to Lemma 2 (cf. (13), (15)) we have

$$|X_n(y)| > \tfrac{1}{2} w(n) \qquad \text{for} \quad y \in (\alpha, \alpha + 1/2n), \qquad (38)$$

and, consequently, the function $X_n(y)$ preserves its sign on the interval $(\alpha, \alpha + 1/2n)$ as a real, continuous function. This means that (cf. (36)–(38), (27))

$$\begin{aligned} |F_n f_n^{(i_0)}(x)| &= \frac{1}{2\pi} \left| \int_{C_n^{(i_0)}} X_n(u-x)\, du \right| \\ &> \frac{1}{2\pi} \cdot \frac{1}{2} w(n) \cdot \mu C_n^{(i_0)} \\ &= \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n}. \end{aligned}$$

Now (35) is proved.

We introduce the function (cf. (28), (24))

$$\Phi_n^{(t)}(x) = \sum_{j=j_0}^{2n} r_j(t)\, f_n^{(j)}(x), \qquad x \in [0, 2\pi), \qquad t \in (0, 1), \qquad (39)$$

where $\{r_j(t)\}_{j=j_0}^{2n}$ are the Rademacher functions.

The following easily verifiable fact is well known (cf., for example, [1, p. 10]): Let $\sum_{j=1}^{m} a_j r_j(t)$ be an arbitrary polynomial in the Rademacher system and i_0 be a fixed natural number, $1 \leqslant i_0 \leqslant m$. Then

$$\mu \left\{ t \in (0, 1) : a_{i_0} r_{i_0}(t) \cdot \sum_{j=1,\, j \neq i_0}^{m} a_j r_j(t) \geqslant 0 \right\} \geqslant \tfrac{1}{2}. \qquad (40)$$

Let (cf. (33), (39))

$$Q = \left\{ (x, t) \in B_n \times (0, 1) : |F_n \Phi_n^{(t)}(x)| > \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n} \right\}. \qquad (41)$$

Then according to (41), (33), (35), and (40), we conclude that for all $x \in B_n$ we have the inequality

$$\int_0^1 X_Q(x, t)\, dt \geqslant \tfrac{1}{2},$$

where X_Q is the characteristic function of the set Q.

Therefore (cf. (34))

$$\int_{B_n} \int_0^1 X_Q(x, t)\, dt \geqslant \tfrac{1}{2} \cdot \mu B_n \geqslant \tfrac{1}{8}.$$

Consequently, by Fubini's theorem there exists a number $t_0 \in (0, 1)$ such that

$$\int_{B_n} X_Q(x, t_0)\, dx \geqslant \tfrac{1}{8}. \tag{42}$$

Relation (42) means that (cf. (41))

$$\mu \left\{ x \in B_n : |F_n \Phi_n^{(t_0)}(x)| > \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n} \right\} \geqslant \frac{1}{8}. \tag{43}$$

Let

$$A_n := \left\{ x \in B_n : |F_n \Phi_n^{(t_0)}(x)| > \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n} \right\} \Big\backslash \bigcup_{j=0}^{2n} C_n^{(j)}. \tag{44}$$

Then it is clear that (cf. (43), (44), (27), (17)) for $n > n_0$

$$\mu A_n \geqslant \frac{1}{8} - \sum_{j=0}^{2n} \mu C_n^{(j)} = \frac{1}{8} - \frac{2n+1}{2n\sqrt{w(n)}} \geqslant \frac{1}{16} \tag{45}$$

and (cf. (16))

$$A_n \subset E_n. \tag{46}$$

According to (39), (16), (27), (28) we have

$$\Phi_n^{(t_0)}(x) = 0 \qquad (x \in E_n) \tag{47}$$

and (cf. (44))

$$|F_n \Phi_n^{(t_0)}(x)| > \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n} \qquad (x \in A_n). \tag{48}$$

Introduce the sets $(j=j_0, ..., 2n)$

$$\Omega_n^{(j)} := \left(\frac{2\pi j}{2n+1} + \mathscr{E}_n, \frac{2\pi j}{2n+1} + \frac{1}{2n\sqrt{w(n)}} - \mathscr{E}_n\right), \tag{49}$$

$$K_n^{(j)} := \left[\frac{2\pi j}{2n+1}, \frac{2\pi j}{2n+1} + \mathscr{E}_n\right], \tag{50}$$

$$Q_n^{(j)} := \left[\frac{2\pi j}{2n+1} + \frac{1}{2n\sqrt{w(n)}} - \mathscr{E}_n, \frac{2\pi j}{2n+1} + \frac{1}{2n\sqrt{w(n)}}\right], \tag{51}$$

$$R_n^{(j)} := \left(\frac{2\pi j}{2n+1} + \frac{1}{2n\sqrt{w(n)}}, \frac{2\pi(j+1)}{2n+1}\right), \tag{52}$$

where

$$\mathscr{E}_n := \frac{1}{64n^2\sqrt{w(n)}}. \tag{53}$$

Now we define the piecewise linear function $g_n \in C_{2\pi}$ by the following equality (cf. (49)–(52))

$$g_n(x) := \begin{cases} \Phi_n^{(t_0)}(x) & \text{for } x \in \bigcup_{j=j_0}^{2n} \Omega_n^{(j)}, \\ 0 & \text{for } x \in \left[0, \frac{2\pi j_0}{2n+1}\right] \cup \left(\bigcup_{j=j_0}^{2n} R_n^{(j)}\right), \\ \text{linear} & \text{on } K_n^{(j)} \text{ and } Q_n^{(j)} \quad (j=j_0, ..., 2n). \end{cases} \tag{54}$$

It is obvious that (cf. (54), (28), (29), (39))

$$\|g_n\|_c \leqslant 1 \tag{55}$$

and (cf. (2), (48), (25), (28), (39), (54), (50), (51), (53), (15)) for $x \in A_n$

$$\begin{aligned} |F_n g_n(x)| &\geqslant |F_n \Phi_n^{(t_0)}(x)| - |F_n(\Phi_n^{(t_0)}(x) - g_n(x))| \\ &\geqslant \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n} - \frac{1}{\pi} \cdot \int_{\bigcup_{j=j_0}^{2n}(K_n^{(j)} \cup Q_n^{(j)})} |X_n(x-u)| \, du \\ &\geqslant \frac{1}{8\pi} \cdot \frac{\sqrt{w(n)}}{n} - \frac{1}{\pi} w(n) \cdot 4\mathscr{E}_n \cdot n \\ &\geqslant \frac{1}{16\pi} \cdot \frac{\sqrt{w(n)}}{n}. \end{aligned} \tag{56}$$

According to (54), (27), (16), and (49)–(51)

$$g_n(x) = 0 \qquad (x \in E_n). \tag{57}$$

Then we find a real-valued trigonometric polynomial $P_n(x)$ such that

$$\|g_n - P_n\|_c \leqslant \min\left(\frac{1}{n \cdot w(n)}, \frac{1}{16n\sqrt{w(n)}}\right). \tag{58}$$

Taking account of (58), (55), (56), (15), (17), and (57) we obtain for $n > n_0$

$$\|P_n\|_c \leqslant 2, \tag{59}$$

$$|F_n P_n(x)| > \frac{1}{32\pi} \cdot \frac{\sqrt{w(n)}}{n} \qquad (x \in A_n) \tag{60}$$

$$|P_n(x)| \leqslant \frac{1}{n \cdot w(n)} \qquad (x \in E_n). \tag{61}$$

Consequently (cf. (17), (59)–(61), (45), (46), (18)–(22)), Lemma 3 is proved.

Now we begin with the proof of the theorem. By induction we define a sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ increasing at infinity. Let $n_1 = n_0 + 1$, where n_0 is the number appearing in Lemma 2.

Now let the numbers $n_1, n_2, ..., n_{k-1}$ be already defined. Consider the function (cf. (15))

$$\alpha_k^{(1)}(x, y_1, ..., y_{k-1}) = \sum_{j=1}^{k-1} \frac{1}{\sqrt[4]{w(n_j)}} \cdot P_{n_j}(x + y_j), \tag{62}$$

where $x \in R$, $y_j \in R$, $j = 1, 2, ..., k-1$, and P_{n_j}, $j = 1, 2, ..., k-1$, are the polynomials appearing in Lemma 3.

For any fixed real numbers y_j, $j = 1, 2, ..., k-1$, for $\alpha_k^{(1)}$ as for the function of x we have (cf. (2), (62))

$$\begin{aligned}\|F_n \alpha_k^{(1)} - \alpha_k^{(1)}\|_c &= \left\| \sum_{j=1}^{k-1} \frac{1}{\sqrt[4]{w(n_j)}} F_n(P_n(\cdot + y_j))(x) - P_{n_j}(x + y_j) \right\|_c \\ &\leqslant \sum_{j=1}^{k-1} \frac{1}{\sqrt[4]{w(n_j)}} \cdot \|F_n P_{n_j}(x) - P_{n_j}(x)\|_c.\end{aligned}$$

Note that for any j, $j=1, 2, ..., k-1$, the polynomial $P_{n_j}(x)$ may be written in the form

$$P_{n_j}(x)=\sum_{m=-m_j}^{m_j} a_m e^{imx} \qquad (j=1, 2, ..., k-1).$$

Consequently, for $n>m_j$ (cf. (4), (6)), $j=1, 2, ..., k-1$,

$$\|F_n P_{n_j}(x)-P_{n_j}(x)\|_c=\left\|\sum_{m=-m_j}^{m_j} a_m \rho_{m,n} e^{imx}-\sum_{m=-m_j}^{m_j} a_m e^{imx}\right\|_c$$
$$\leqslant \sum_{m=-m_j}^{m_j} |a_m|\, O_m\left(\frac{1}{n}\right)=O_j\left(\frac{1}{n}\right).$$

From the last inequality we conclude that when $n>\max\{m_j, j=1, 2, ..., k-1\}$

$$\|F_n \alpha_k^{(1)}-\alpha_k^{(1)}\|_c \leqslant \sum_{j=1}^{k-1} \frac{1}{\sqrt[4]{\omega(n_j)}}\cdot O_j\left(\frac{1}{n}\right)\leqslant \frac{M_k}{n} \tag{63}$$

and (cf. (4)) analogously

$$\|T_n \alpha_k^{(1)}-\alpha_k^{(1)}\|_c \leqslant M_k/n \tag{64}$$

for all $y_j \in R$, $j=1, 2, ..., k-1$, where M_k is a positive constant independent of n and y_j, $j=1, 2, ..., k-1$.

Now we define the index n_k with $n_k>n_{k-1}$, $n_k>\max\{m_j, j=1, ..., k-1\}$ such that the inequalities (cf. (10), (15), (63))

$$\frac{1}{\omega(n_k)}\leqslant\frac{1}{16}\cdot\frac{1}{\omega(n_{k-1})} \tag{65}$$

$$\frac{1}{n_k\cdot\sqrt[4]{\omega(n_k)}}\leqslant\frac{1}{2}\cdot\frac{1}{n_{k-1}\cdot\sqrt[4]{\omega(n_{k-1})}}, \tag{66}$$

$$\frac{M_k}{n_k}\leqslant\frac{\sqrt[4]{\omega(n_k)}}{kn_k}, \tag{67}$$

$$\omega(n_{k-1})\frac{4}{\sqrt[4]{\omega(n_k)}}\leqslant\frac{\sqrt[4]{\omega(n_{k-1})}}{k\cdot n_{k-1}}, \tag{68}$$

$$\frac{4}{\sqrt[4]{\omega(n_k)}}\leqslant\frac{\sqrt[4]{\omega(n_{k-1})}}{k\cdot n_{k-1}} \tag{69}$$

hold.

Thus we have obtained an infinite increasing sequence of indices $\{n_k\}_{k=1}^{\infty}$.

Now we use the theorem of N. Kirchoff and R. J. Nessel (cf. (8)). Instead of the set H_k we take 2π-periodic extension $A_{n_k}^*$ of the set A_{n_k} from the Lemma 3 corresponding to the number n_k. As a set D_k we take the set

$$D_k := \bigcup_{j=0}^{2n_k} \left(\frac{2\pi j}{2n_k+1} + \frac{1}{2n_k \cdot \sqrt{\omega(n_k)}}, \frac{2\pi(j+1)}{2n_k+1} - \frac{1}{2n_k \sqrt{\omega(n_k)}} \right). \tag{70}$$

We see (cf. (70), (10), (15))

$$\mu D_k = (2n_k+1) \cdot \left(\frac{2\pi}{2n_k+1} - \frac{1}{n_k \sqrt{\omega(n_k)}} \right)$$
$$= 2\pi - o(1) \qquad (k \to \infty).$$

Consequently, for all $t \in (0, 2\pi)$ (cf. (20), (70))

$$\sum_{k=1}^{\infty} \frac{\mu(D_k \cap (A_{n_k}^* - t))}{\mu D_k} \geqslant \sum_{k=1}^{\infty} \frac{\gamma_1 - o(1)}{2\pi} = +\infty$$

and thus, condition (8) holds. From the theorem of N. Kirchoff and R. J. Nessel we conclude that there exist points $y_k^{(0)} \in D_k$, $k = 1, 2, \ldots$, such that the set

$$A := \limsup_{k \to \infty} (A_{n_k}^* - y_k^{(0)}) \tag{71}$$

is a set of full measure.

We introduce the functions $(k = 1, 2, \ldots)$

$$\varphi_k(x) := P_{n_k}(x + y_k^{(0)}) \qquad (x \in R). \tag{72}$$

Now we shall show that for all $k = 1, 2, \ldots$, and $j = 0, 1, \ldots, 2n_k$ we have

$$\frac{2\pi j}{2n_k+1} + y_k^{(0)} \in E_{n_k}^*, \tag{73}$$

where $E_{n_k}^*$ is 2π-periodic extension of the set E_{n_k} (cf. (16)).

Indeed, $y_k^{(0)} \in D_k$ means that (cf. (70)) for some j_1, $0 \leqslant j_1 \leqslant 2n_k$, one has

$$\frac{1}{2n_k \sqrt{w(n_k)}} + \frac{2\pi j_1}{2n_k+1} < y_k^{(0)} < \frac{2\pi(j_1+1)}{2n_k+1} - \frac{1}{2n_k \sqrt{\omega(n_k)}}.$$

Consequently,

$$\frac{1}{2n_k\sqrt{\omega(n_k)}}+\frac{2\pi(j+j_1)}{2n_k+1}<y_k^{(0)}+\frac{2\pi j}{2n_k+1}<\frac{2\pi(j+j_1+1)}{2n_k+1}-\frac{1}{2n_k\sqrt{\omega(n_k)}}.$$

Dividing the number $j+j_1$ by $2n_k+1$ we obtain

$$j+j_1=(2n_k+1)\,q_k+r_k \qquad (0\leqslant r_k\leqslant 2n_k),$$

where q_k and r_k are nonnegative integers. Therefore

$$\frac{1}{2n_k\sqrt{\omega(n_k)}}+2\pi q_k+\frac{2\pi r_k}{2n_k+1}<y_k^{(0)}+\frac{2\pi j}{2n_k+1}$$
$$<2\pi q_k+\frac{2\pi(r_k+1)}{2n_k+1}-\frac{1}{2n_k\sqrt{\omega(n_k)}}.$$

The last inequality means that (cf. (16))

$$y_k^{(0)}+\frac{2\pi j}{2n_k+1}\in E_{n_k}+2\pi q_k\subset E^*_{n_k}.$$

Consequently (73) is proved.
From (73) it follows that (cf. (22), (72))

$$\left|\varphi_k\left(\frac{2\pi j}{2n_k+1}\right)\right|\leqslant\frac{1}{n_k\omega(n_k)},\qquad j=0,1,\ldots,2n_k,\qquad k=1,2,\ldots. \tag{74}$$

Consider the functions (cf. (72))

$$f(x):=\sum_{j=1}^{\infty}\frac{1}{\sqrt[4]{\omega(n_j)}}\cdot\varphi_j(x), \tag{75}$$

$$\gamma_k(x):=\sum_{j=k+1}^{\infty}\frac{1}{\sqrt[4]{\omega(n_j)}}\cdot\varphi_j(x), \tag{76}$$

$$\alpha_k(x):=\sum_{j=1}^{k-1}\frac{1}{\sqrt[4]{\omega(n_j)}}\cdot\varphi_j(x). \tag{77}$$

It is clear that (cf. (75), (76), (18), (65), (72))

$$\|f\|_c \leqslant 2 \cdot \sum_{j=1}^{\infty} \frac{1}{\sqrt[4]{\omega(n_j)}} < +\infty \tag{78}$$

$$\|\gamma_k\|_c \leqslant 4 \cdot \frac{1}{\sqrt[4]{\omega(n_{k+1})}}. \tag{79}$$

Let $x \in A$ (cf. (71)). Then for infinitely many indices k we have

$$x \in A^*_{n_k} - y_k^{(0)}. \tag{80}$$

Fix any such k. We have (cf. (80))

$$x = a_k + 2\pi \cdot l_k - y_k^{(0)} \qquad (l_k \in Z,\ a_k \in A_{n_k}). \tag{81}$$

Therefore (cf. (72), (2), (81))

$$\begin{aligned} |F_{n_k}\varphi_k(x) - \varphi_k(x)| &= |F_{n_k}P_{n_k}(a_k) - P_{n_k}(a_k)| \\ &\geqslant \frac{C_1\sqrt{\omega(n_k)}}{n_k}. \end{aligned} \tag{82}$$

According to (2), (75)–(77), (82), (67)–(69), (63), (62), (72), (17), (10), (15), and (79) we obtain

$$\begin{aligned} |F_{n_k}f(x) - f(x)| &\geqslant \frac{1}{\sqrt[4]{\omega(n_k)}}\,|F_{n_k}\varphi_k(x) - \varphi_k(x)| \\ &\quad - |F_{n_k}\alpha_k(x) - \alpha_k(x)| - |F_{n_k}\gamma_k(x)| - |\gamma_k(x)| \\ &\geqslant \frac{1}{\sqrt[4]{\omega(n_k)}} \cdot \frac{C_1\sqrt{\omega(n_k)}}{n_k} - \frac{\sqrt[4]{\omega(n_k)}}{k \cdot n_k} \\ &\quad - \omega(n_k) \cdot \|\gamma_k\|_c - \|\gamma_k\|_c \\ &\geqslant \frac{C_1\sqrt[4]{\omega(n_k)}}{n_k} - \frac{\sqrt[4]{\omega(n_k)}}{k \cdot n_k} - \omega(n_k) \cdot \frac{4}{\sqrt[4]{\omega(n_{k+1})}} - \frac{4}{\sqrt[4]{\omega(n_{k+1})}} \\ &\geqslant C_1 \cdot \frac{\sqrt[4]{\omega(n_k)}}{n_k} - \frac{\sqrt[4]{\omega(n_k)}}{k \cdot n_k} - \frac{\sqrt[4]{\omega(n_k)}}{(k+1)\cdot n_k} - \frac{\sqrt[4]{\omega(n_k)}}{(k+1)\cdot n_k} \\ &= \frac{\sqrt[4]{\omega(n_k)}}{n_k} \cdot \left(C_1 - \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+1}\right) \\ &= \frac{\sqrt[4]{\omega(n_k)}}{n_k} \cdot (C_1 - o(1)) \qquad (k \to \infty). \end{aligned} \tag{83}$$

From (1), (75)–(77), (3), (10), (15), (72), (63), (74), (72), (67), (81), (19), (22), (79), (68), and (69) one has

$$\begin{aligned}
|T_{n_k}f(x)-f(x)| &\leqslant \frac{1}{\sqrt[4]{\omega(n_k)}}\cdot(|T_{n_k}\varphi_k(x)|+|\varphi_k(x)|)\\
&\quad+|T_{n_k}\alpha_k(x)-\alpha_k(x)|+|T_{n_k}\gamma_k(x)|+|\gamma_k(x)|\\
&\leqslant \frac{1}{\sqrt[4]{\omega(n_k)}}\cdot\left(\frac{\omega(n_k)}{2n_k+1}\cdot\sum_{j=0}^{2n_k}\left|\varphi_k\left(\frac{2\pi j}{2n_k+1}\right)\right|+|P_{n_k}(x+y_k^{(0)}|\right)\\
&\quad+\frac{\sqrt[4]{\omega(n_k)}}{k\cdot n_k}+\omega(n_k)\cdot\|\gamma_k\|_c+\|\gamma_k\|_c\\
&\leqslant \frac{1}{\sqrt[4]{\omega(n_k)}}\cdot\left(\frac{1}{n_k}+\frac{1}{n_k\cdot\omega(n_k)}\right)+\frac{\sqrt[4]{\omega(n_k)}}{k\cdot n_k}\\
&\quad+\omega(n_k)\cdot\frac{4}{\sqrt[4]{\omega(n_{k+1})}}+\frac{4}{\sqrt[4]{\omega(n_{k+1})}}\\
&\leqslant \frac{1}{n_k\cdot\sqrt[4]{\omega(n_k)}}+\frac{1}{n_k\omega(n_k)\cdot\sqrt[4]{\omega(n_k)}}+\frac{\sqrt[4]{\omega(n_k)}}{k\cdot n_k}\\
&\quad+\frac{\sqrt[4]{\omega(n_k)}}{(k+1)\,n_k}+\frac{\sqrt[4]{\omega(n_k)}}{(k+1)\cdot n_k}\\
&=\frac{\sqrt[4]{\omega(n_k)}}{n_k}\cdot\left(\frac{1}{\sqrt[4]{\omega(n_k)}}+\frac{1}{\omega(n_k)\cdot\sqrt[4]{\omega(n_k)}}+\frac{1}{k}+\frac{2}{k+1}\right)\\
&=\frac{\sqrt[4]{\omega(n_k)}}{n_k}\cdot o(1)\qquad(k\to\infty).
\end{aligned}\tag{84}$$

Consequently (cf. (71), (78), (9), (83), and (84)) the theorem is proved.

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